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Positive Values of Non-homogeneous Indefinite Quadratic Forms of Type (2, 5)

RANJEET SEHMI

Department of Applied Sciences, Panjab Engineering College, Chandigarh, 160012, India

AND

V. C. DUMIR

Centre for Advanced Study in Mathematics, Panjab University, Chandigarh, 160014, India

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The minimum $\Gamma_{r,n-r}$ of positive values of non-homogeneous indefinite quadratic forms of type $(r, n-r)$ is defined as the infimum of all constants $\Gamma > 0$ such that for any indefinite quadratic form Q of type $(r, n-r)$ and determinant $D \neq 0$ and any real numbers $c_1, ..., c_n$ there exist integers $x_1, ..., x_n$ such that

$$0 < Q(x_1 + c_1, ..., x_n + c_n) < (\Gamma |D|)^{1/n}.$$

In this paper it is proved that $\Gamma_{2,5} = 32$, thereby confirming the conjecture of Bambah, Dumir, and Hans-Gill. Also, all the critical forms for which equality is needed are determined.

1. INTRODUCTION

Let $Q(x_1, ..., x_n)$ be a real indefinite quadratic form in n variables, of type $(r, n-r)$ and determinant $D \neq 0$. Blaney [9] has shown that there exist constants Γ, independent of Q and depending only on n and r, such that given any real numbers $c_1, ..., c_n$ there exist $(x_1, ..., x_n) \equiv (c_1, ..., c_n) \pmod 1$ such that

$$0 < Q(x_1, ..., x_n) \leqslant (\Gamma |D|)^{1/n}.$$

Let $\Gamma_{r,n-r}$ denote the infimum of all such numbers Γ.

In this notation the Davenport–Heilbronn result [11] implies $\Gamma_{1,1} = 4$. Blaney [10] and Barnes [7] independently proved $\Gamma_{2,1} = 4$. $\Gamma_{1,2}$, $\Gamma_{2,2}$, $\Gamma_{3,1}$

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were evaluated by Dumir [12, 13]. Dumir and Hans-Gill [14] determined $\Gamma_{1,3}$. Hans-Gill and Madhu Raka [15, 16] evaluated $\Gamma_{4,1}$ and $\Gamma_{3,2}$. $\Gamma_{r,n-r}$ for forms of signature $s=2r-n=0, \pm 1, 2, 3$ for all n, are due to Bambah, Dumir, and Hans-Gill [4–6]. Aggarwal and Gupta [1–3] have determined $\Gamma_{r,r+2}$ and $\Gamma_{r,r+3}$ for $r \geqslant 3$ and $\Gamma_{r,r+4}$ for $r \geqslant 1$. Dumir, Hans-Gill, and Woods have proved that for $n \geqslant 6$, $\Gamma_{r,n-r}$ depends only on n and signature $s \pmod 8$ (unpublished). Thus except for $\Gamma_{2,5}$, $\Gamma_{2,4}$, and $\Gamma_{1,4}$ all values of $\Gamma_{r,n-r}$ have been obtained. It may be remarked here that for larger values of n the evaluation of $\Gamma_{r,n-r}$ is relatively easy. For small values of n, detailed analysis and careful investigation is needed. In this paper, we prove that $\Gamma_{2,5}=32$, thereby proving the conjecture of Bambah, Dumir, and Hans-Gill in this case. More precisely, we prove:

THEOREM. *Let $Q(x_1, ..., x_7)$ be a real indefinite quadratic form of the type $(2, 5)$ and determinant $D \neq 0$. Then given any real numbers $c_1, ..., c_7$ there exist $(x_1, ..., x_7) \equiv (c_1, ..., c_7) \pmod 1$ such that*

$$0 < Q(x_1, ..., x_7) \leqslant (32\,|D|)^{1/7}. \tag{1.1}$$

Moreover, equality is necessary in (1.1) iff Q is equivalent to ρQ_1, ρQ_2, or ρQ_3 and $(c_1, ..., c_7)$ is equivalent to the points P_1, P_2, or P_3, respectively, where $\rho > 0$ and

$$\begin{aligned} Q_1 &= x_1^2+x_2^2-x_3^2-x_4^2-2(x_5^2+x_6^2+x_7^2+x_5x_6+x_5x_7+x_6x_7),\\ P_1 &= (1/2, 1/2, 1/2, 1/2, 0, 0, 0), \end{aligned} \tag{1.2}$$

$$\begin{aligned} Q_2 &= x_1x_2+x_3x_4-(x_5^2+x_6^2+x_7^2+x_5x_6+x_5x_7+x_6x_7),\\ P_2 &= (0, ..., 0), \end{aligned} \tag{1.3}$$

$$\begin{aligned} Q_3 &= x_1^2-x_2^2+4x_3x_4-(x_4^2+x_5^2+x_6^2+x_7^2),\\ P_3 &= (1/2, ..., 1/2). \end{aligned} \tag{1.4}$$

2. SOME LEMMAS

In the course of the proof we use the following lemmas.

LEMMA 1. *Let $Q(x_1, ..., x_n)$ be a real incommensurable indefinite quadratic form of determinant $D \neq 0$, in $n \geqslant 3$ variables. Then for any $\varepsilon > 0$, there exist $x_1, ..., x_n \in \mathbb{Z}$, not all zero, such that*

$$|Q(x_1, ..., x_n)| < \varepsilon. \tag{2.1}$$

This is a result of Margulis [19].

LEMMA 2. *Let $Q(x_1, \ldots, x_n)$ be a real indefinite quadratic form in $n \geqslant 3$ variables with determinant $D \neq 0$ such that Q takes arbitrarily small values for integers $x_1, \ldots, x_n$. Let $c_1, \ldots, c_n$, α, δ be real numbers with $\delta > 0$. Then we can find $(x_1, \ldots, x_n) \equiv (c_1, \ldots, c_n)$ (mod 1) such that*

$$\alpha - \delta < Q(x_1, \ldots, x_n) < \alpha + \delta. \tag{2.2}$$

This is a result of Watson [21].

LEMMA 3. *Let $Q(x_1, \ldots, x_n)$ be a rational non-singular indefinite form in $n \geqslant 5$ variables. Then it is a zero form.*

This is well known as Meyer's Theorem.

LEMMA 4. *If $Q(x_1, \ldots, x_n)$ is a rational indefinite form of the type $(2, 5)$ then*

$$\begin{aligned} Q \sim \rho[&(x_1 + a_2x_2 + a_4x_4 + a_5x_5 + a_6x_6 + a_7x_7)x_2 \\ &+ m(x_3 + b_4x_4 + b_5x_5 + b_6x_6 + b_7x_7)x_4 - \phi(x_5, x_6, x_7)], \end{aligned} \tag{2.3}$$

where $m \geqslant 1$ is an integer, $-1/2 < a_i \leqslant 1/2$, $-1/2 < b_i \leqslant 1/2$, $\rho > 0$, and ϕ is a positive definite ternary form.

Further, if $a_2 = 0$ then $a_i = 0$ $\forall i$ and if $b_4 = 0$ then $b_j = 0$ $\forall j$. If $m = 1$ and $b_4 = 0$ then $a_4 = 0$.

This follows from the proof of Lemma 12 of Birch [8].

We use the following conventions.

If for any given real numbers $c_1, \ldots, c_n$ there exist $(x_1, \ldots, x_n) \equiv (c_1, \ldots, c_n)$ (mod 1) satisfying

$$\alpha < Q(x_1, \ldots, x_n) < \beta, \tag{2.4}$$

then we say that (2.4) is soluble. If there are integers $u_1, \ldots, u_n$ such that $Q(u_1, \ldots, u_n) = \gamma \neq 0$, then we say that Q represents γ.

LEMMA 5. *Let $Q(x_1, \ldots, x_n)$ be a zero form of determinant $D \neq 0$. Let α, β be real numbers satisfying*

$$\beta - \alpha > 2\,|D|^{1/n}.$$

Then

$$\alpha < Q(x_1, \ldots, x_n) < \beta \tag{2.5}$$

is soluble.

This is a result of Jackson [17].

LEMMA 6. *Let α, β, γ be real numbers with $\gamma > 1$. Let m be the integer defined by $m < \gamma \leqslant m+1$. Then for any real x_0 there exists $x \equiv x_0 \pmod 1$ such that*

$$0 < (x+\alpha)^2 + \beta < \gamma, \tag{2.6}$$

provided

$$-m^2/4 < \beta < \gamma - 1/4.$$

This is a result of Dumir [13].

LEMMA 7. *Let α, β, γ be real numbers with $\gamma > 1$. Let m be the integer defined by $m < \gamma \leqslant m+1$. Then for any real x_0 there exists $x \equiv x_0 \pmod 1$ such that*

$$0 < -(x+\alpha)^2 + \beta < \gamma, \tag{2.7}$$

provided

$$1/4 < \beta < m^2/4 + \gamma.$$

This is a result of Dumir [12].

LEMMA 8. *Let $\phi(x, y, z)$ be a positive definite quadratic form of determinant $D \neq 0$. Then there exist integers x, y, z such that*

$$0 < \phi(x, y, z) \leqslant (2\,|D|)^{1/3},$$

with equality iff $\phi \sim \rho(x^2 + y^2 + z^2 + xy + yz + zx)$, $\rho > 0$.

This is a result of Gauss and Seeber.

LEMMA 9. *If $\psi(x, y)$ is a positive definite binary quadratic form of determinant δ then*

$$\psi(x, y) \sim Ax^2 + Bxy + Cy^2, \tag{2.8}$$

where

$$0 \leqslant B \leqslant A \leqslant C \qquad \text{and} \qquad AC \leqslant 4\delta/3.$$

LEMMA 10. *Let α, β, γ be real numbers with $\alpha > 0$, $\gamma > 0$. Let $2h$, k be integers such that*

$$|h\alpha - k^2/\alpha^2| + \alpha/2 \leqslant \gamma. \tag{2.9}$$

Suppose that either $\alpha^3 \neq k^2/h$ *or* $\beta \not\equiv \alpha h/k \pmod{\alpha/k,\ 2/\alpha^2}$.[1] *Then for any real number* v, *there exist integers* x, y *satisfying*

$$0 < \alpha x + \beta y - y^2/\alpha^2 + v \leqslant \gamma. \tag{2.10}$$

This is Lemma 6 of Macbeath [18]. It is easy to see that strict inequality in (2.9) implies strict inequality in (2.10). It can also be seen that the hypotheses of Lemma 10 imply that the inequalities

$$0 < \pm\alpha x + \beta y + y^2/\alpha^2 + v < \gamma$$

are soluble in integers x and y. This lemma can be restated in the following convenient form for the purpose of application:

LEMMA 11. *Let* α, β, γ *be real numbers with* $\alpha > 0$, $\gamma > 0$. *Let* $2h$, k *be integers such that*

$$|h - k^2\alpha| + 1/2 < \gamma. \tag{2.11}$$

Suppose that either $\alpha \neq h/k^2$ *or* $\beta \not\equiv (h/k) \bmod (1/k, 2\alpha)$. *Then for any real number* v, *there exist integers* x, y *satisfying*

$$0 < \pm x + \beta y \pm \alpha y^2 + v < \gamma. \tag{2.12}$$

3. REDUCTION

3.1. If Q is an incommensurable quadratic form, then the result follows from Lemma 1 and Lemma 2. So we can suppose that Q is a rational indefinite quadratic form of determinant $D \neq 0$. By Lemma 3, it is a zero form. Let $d = (32\,|D|)^{1/7}$.

LEMMA 12. *If* Q *represents a number* a *with* $0 < |a| < d/3$ *then the inequality*

$$0 < Q(x_1, ..., x_7) < d \tag{3.1}$$

is soluble.

Proof. Without loss of generality we can suppose that Q represents a primitively. On replacing Q by an equivalent form we can suppose that $Q(1, 0, ..., 0) = a$ and write

$$Q(x_1, ..., x_7) = a(x_1 + \cdots)^2 + \phi(x_2, ..., x_7), \tag{3.2}$$

[1] $\beta - \alpha h/k$ is not an integral combination of α/k and $2/\alpha^2$.

ϕ is an indefinite form in 6 variables. Hence, it is a zero form. By homogeneity we can suppose that $|a| = 1$, so that $d > 3$. Let n be the integer such that $n < d \leqslant n+1$, then $n \geqslant 3$.

First suppose that $a = 1$.

By Lemma 6, the inequality

$$0 < (x_1 + \cdots)^2 + \phi(x_2, ..., x_7) < d \tag{3.3}$$

is soluble if we can solve

$$-n^2/4 < \phi(x_2, ..., x_7) < d - 1/4. \tag{3.4}$$

Since ϕ is a zero form of determinant D, by Lemma 5, (3.4) is soluble if

$$d + (n^2 - 1)/4 > 2\,|D|^{1/6} = (2d^7)^{1/6}, \tag{3.5}$$

i.e., if

$$f(d) = \frac{d + (n^2 - 1)/4}{d^{7/6}} > 2^{1/6}. \tag{3.6}$$

For a fixed n, $f(d)$ is a decreasing function of d and $d \leqslant n+1$, so that

$$f(d) \geqslant f(n+1) = (n+3)/4(n+1)^{1/6} = g(n), \qquad \text{say.} \tag{3.7}$$

Since $g(n)$ is an increasing function of n and $n \geqslant 3$, therefore

$$g(n) \geqslant g(3) = 3/2(4)^{1/6} > 2^{1/6}.$$

Hence, (3.6) holds and the lemma follows when $a = 1$.

Now suppose that $a = -1$.

By Lemma 7

$$0 < -(x_1 + \cdots)^2 + \phi(x_2, ..., x_7) < d \tag{3.8}$$

is soluble if we can solve

$$1/4 < \phi(x_2, ..., x_7) < d + n^2/4.$$

Since ϕ is a zero form of determinant $-D$, (3.8) is soluble by Lemma 5, if

$$d + (n^2 - 1)/4 > 2\,|D|^{1/6} = (2d^7)^{1/6}. \tag{3.9}$$

This inequality is the same as (3.5) which has been verified above. This completes the proof of the lemma.

3.2. Let $Q(x_1, ..., x_7)$ be a rational quadratic form of the type (2, 5). On replacing Q by an equivalent form by Lemma 4 and by homogeneity we can suppose that

$$\begin{aligned} Q = {} & (x_1 + a_2x_2 + a_4x_4 + a_5x_5 + a_6x_6 + a_7x_7)x_2 \\ & + m(x_3 + b_4x_4 + b_5x_5 + b_6x_6 + b_7x_7)x_4 - \phi(x_5, x_6, x_7), \end{aligned}$$

where $m \geqslant 1$ is an integer and ϕ is a positive definite ternary form with determinant $\Delta = (16\,|D|)/m^2$. Let

$$a = \min(\phi(X) \colon X \in Z^3,\ X \neq 0).$$

By Lemma 8, we have

$$0 < a \leqslant (2\Delta)^{1/3} = (32\,|D|/m^2)^{1/3} = (d^7/m^2)^{1/3}. \tag{3.10}$$

Further, by a unimodular transformation we can suppose that $\phi(1, 0, 0) = a$ and write

$$\phi(x_5, x_6, x_7) = a(x_5 + h_6x_6 + h_7x_7)^2 + \psi(x_6, x_7), \tag{3.11}$$

where ψ is a positive definite binary form with determinant δ. By Lemma 9, we can suppose without loss of generality that

$$\psi(x_6, x_7) = Ax_6^2 + Bx_6x_7 + Cx_7^2, \tag{3.12}$$

where $0 \leqslant B \leqslant A \leqslant C$ and

$$A^2 \leqslant AC \leqslant 4\delta/3 = 4\Delta/3a = (64\,|D|)/3am^2 = 2d^7/3am^2. \tag{3.13}$$

Moreover, we can suppose that $-1/2 < h_6 \leqslant 1/2$ and $-1/2 < h_7 \leqslant 1/2$, $-1/2 < a_i,\ b_j \leqslant 1/2$.

Since $A + ah_6^2$ is a value of the form ϕ, therefore

$$a \leqslant A + ah_6^2 \leqslant A + a/4, \tag{3.14}$$

$$\Rightarrow 3a/4 \leqslant A \qquad \text{and} \qquad a \leqslant A \text{ in case } h_6 = 0. \tag{3.15}$$

If $a < d/3$ then it follows from Lemma 12 that (1.1) is satisfied with strict inequality because Q represents $-a$. So we can suppose that $d/3 \leqslant a$.

Therefore

$$d/3 \leqslant a \leqslant (d^7/m^2)^{1/3}, \tag{3.16}$$

$$\Rightarrow d^4 \geqslant m^2/27. \tag{3.17}$$

We want to prove that

$$0 < (x_1 + \cdots)x_2 + m(x_3 + \cdots)x_4 - \phi(x_5, x_6, x_7) < d \tag{3.18}$$

is soluble except when $Q \sim Q_i$, $i = 1, 2, 3$.

Without loss of generality we can suppose that $-1/2 < c_i \leqslant 1/2$. Choose $(x_3, ..., x_7) \equiv (c_3, ..., c_7) \pmod 1$ arbitrarily. Choose $x_2 \equiv c_2 \pmod 1$ satisfying $0 < |x_2| \leqslant 1/2$ if $c_2 \not\equiv 0 \pmod 1$. Otherwise choose $x_2 = 1$. Then choose $x_1 \equiv c_1 \pmod 1$ satisfying

$$0 < (x_1 + \cdots)x_2 + m(x_3 + \cdots)x_4 - \phi(x_5, x_6, x_7) \leqslant |x_2|.$$

If $c_2 \not\equiv 0 \pmod 1$ and $d > |x_2|$ then (3.18) is soluble. In particular (3.18) is soluble if $m \geqslant 2$, because

$$m \geqslant 2 \Rightarrow d^4 \geqslant 4/27, \qquad \text{(by (3.17))}$$
$$\Rightarrow d > 1/2 \geqslant |x_2|.$$

If $c_2 \equiv 0 \pmod 1$ and $d > 1$ then again (3.18) is soluble.

Thus, we are left with the following cases:

(i) $c_2 \not\equiv 0 \pmod 1$, $m = 1$, and $d \leqslant |x_2|$. (3.19)

(ii) $c_2 \equiv 0 \pmod 1$ and $d \leqslant 1$. (3.20)

4. Proof of the Theorem

4.1. $c_2 \not\equiv 0 \pmod 1$, $m = 1$, and $d \leqslant |x_2|$.

Choose $(x_3, x_4, x_6, x_7) \equiv (c_3, c_4, c_6, c_7) \pmod 1$ arbitrarily. Write $x_1 = x + c_1$ and $x_5 = y + c_5$. Then (3.18) is soluble if we can find integers x and y such that

$$0 < xx_2 + \beta' y - ay^2 + v' < d,$$

i.e.,

$$0 < \pm x + \beta y - \alpha y^2 + v < d/|x_2|, \tag{4.1}$$

where β and v are some constants and $\alpha = a/|x_2|$.

Moreover

$$\alpha = a/|x_2| \leqslant a/d \leqslant d^{7/3}/d = d^{4/3} < (1/2)^{4/3} < 1/2$$

and

$$\alpha = a/|x_2| \geqslant 2a \geqslant 2d/3 > 2(1/27)^{1/4}/3 > 1/4.$$

Taking $h=1/2$, $k=1$, $\gamma=d/|x_2|$, it follows from Lemma 11 that (4.1) is soluble if

$$|1/2-a/|x_2|\ |+1/2<d/|x_2|,$$

which is satisfied if $|x_2|<a+d$.

Now suppose that

$$a+d\leqslant|x_2|. \tag{4.2}$$

Again, taking $h=1$, $k=2$, $\gamma=d/|x_2|$, it follows from Lemma 11 that (4.1) is soluble if

$$|1-4a/|x_2|\ |+1/2<d/|x_2|. \tag{4.3}$$

Now $7a<3d$, since $a^3\leqslant d^7\leqslant d^3/16<(3d/7)^3$, so that using (4.2), we have $|x_2|+2d\geqslant a+3d>8a$ and, hence, (4.3) is satisfied and (3.18) is soluble if $c_2\not\equiv 0 \pmod 1$.

4.2. $c_2\equiv 0 \pmod 1$ and $d\leqslant 1$.

LEMMA 13. *Relation* (3.18) *is soluble if*

(i) $a<1/2$ *and* $a+d>1$. *In particular if* $m\geqslant 3$.

(ii) $a=1/2$ *except perhaps when* $h_6=h_7=b_5=0$, $m=1$ *or* 2, *and* $(a_5, c_5)=(0, 1/2)$ *or* $(1/2, 0)$ *if* $m=2$ *and* $(a_5, c_5)=(0, 1/2)$ *if* $m=1$.

(iii) $a>1/2$ *except perhaps when* $Q\sim\rho Q_1$ *and* $(c_1, ..., c_7)=(1/2, 1/2, 1/2, 1/2, 0, 0, 0)$; $Q\sim\rho Q_2$ *and* $(c_1, ..., c_7)=(0, ..., 0)$,

where $\rho>0$ *and* Q_1, Q_2 *are as given in* (1.2) *and* (1.3), *respectively. In both these cases equality is necessary in* (1.1).

Proof. Choose $(x_3, x_4, x_6, x_7)\equiv(c_3, c_4, c_6, c_7) \pmod 1$ arbitrarily, $x_2=\pm 1$. Write $x_1=x+c_1$, $x_5=y+c_5$. Then (3.18) is soluble if we can find integers x and y such that

$$0<\pm x+\beta y-ay^2+v<d, \tag{4.4}$$

where

$$\beta=\pm a_5+mb_5x_4-2ah_6x_6-2ah_7x_7-2ac_5$$

and v is some constant.

Taking $h=1/2$, $k=1$, $\alpha=a$, $\gamma=d$, it follows from Lemma 11 that (4.4) is soluble for $a\neq 1/2$ if

$$|1/2-a|+1/2<d. \tag{4.5}$$

For $a < 1/2$, (4.5) is satisfied if

$$1 < a + d. \tag{4.6}$$

In particular, if $m \geqslant 3$, then

$$a + d \geqslant d/3 + d = 4d/3 > 1 \qquad \text{for} \quad d^4 \geqslant (m^2/27) \geqslant 1/3,$$

so that (4.6) is satisfied. This completes part (i) of the lemma. If $a = 1/2$, then $1/2 \leqslant (d^7/m^2)^{1/3} \leqslant (1/m^2)^{1/3}$, which implies that $m = 1$ or 2.

Taking $h = 1/2$, $k = 1$, $\alpha = 1/2$, $\gamma = d$, it follows from Lemma 11 that (4.4) is soluble for $a = 1/2$ unless

$$\beta \equiv h/k \qquad \mathrm{mod}(1/k, 2\alpha),$$

i.e.,

$$\pm a_5 + mb_5 x_4 - h_6 x_6 - h_7 x_7 - c_5 \equiv 1/2 \qquad (\mathrm{mod}\ 1) \tag{4.7}$$

for all choices of x_4, x_6, and x_7.

Taking $x_6 = c_6$ and $1 + c_6$, we get $h_6 \equiv 0$ (mod 1), since $-1/2 < h_6 \leqslant 1/2$, we have $h_6 = 0$. Similarly $h_7 = 0$ and $mb_5 \equiv 0$ (mod 1).

Since $-1/2 < b_5 \leqslant 1/2$, for $m = 1$ we get $b_5 = 0$ and by symmetry $a_5 = 0$ and hence from (4.7), $c_5 = 1/2$. For $m = 2$ we get $b_5 = 0$ or $1/2$. If $b_5 = 1/2$ then

$$\begin{aligned} Q &= (x_1 + \cdots)\, x_2 + 2(x_3 + b_4 x_4 + x_5/2 + b_6 x_6 + b_7 x_7)\, x_4 \\ &\quad - x_5^2/2 - \psi(x_6, x_7) \\ &= (x_1 + \cdots)\, x_2 + 2(x_3 + b_4' x_4 + b_6 x_6 + b_7 x_7)\, x_4 \\ &\quad - (x_5 - x_4)^2/2 - \psi(x_6, x_7) \\ &\sim (x_1 + \cdots)\, x_2 + 2(x_3 + b_4' x_4 + b_6 x_6 + b_7 x_7)\, x_4 \\ &\quad - x_5^2/2 - \psi(x_6, x_7). \end{aligned}$$

Therefore, replacing Q by an equivalent form we can suppose that $b_5 = 0$ and then (4.7) reduces to

$$\begin{aligned} &\pm a_5 - c_5 \equiv (1/2) \qquad (\mathrm{mod}\ 1) \\ &\Rightarrow (a_5, c_5) = (0, 1/2) \qquad \text{or} \quad (1/2, 0). \end{aligned} \tag{4.8}$$

This completes part (ii) of the lemma.

Now suppose that $a > 1/2$, so that (4.5) is satisfied if $a < d$. We have

$$a \leqslant (2\Delta)^{1/3} = (d^7/m^2)^{1/3} \leqslant d^{7/3} \leqslant d. \tag{4.9}$$

Therefore, $a < d$ unless equality holds throughout in (4.9). That is, (4.5) is satisfied unless $d = m = 1$ and $a = (2\Delta)^{1/3} = d^{7/3} = 1$.

By Lemma 8, $a = (2\Delta)^{1/3}$ implies $\phi \sim \rho(x_5^2 + x_6^2 + x_7^2 + x_5x_6 + x_5x_7 + x_6x_7)$. Since $d = 1$, therefore $1 = d^7 = 32\,|D| = \rho^3$ implies $\rho = 1$. Without loss of generality suppose that $\phi = (x_5^2 + x_6^2 + x_7^2 + x_5x_6 + x_5x_7 + x_6x_7)$, so that

$$Q = (x_1 + \cdots)\,x_2 + (x_3 + \cdots)\,x_4 - (x_5^2 + x_6^2 + x_7^2 + x_5x_6 + x_5x_7 + x_6x_7).$$

By symmetry we have $c_4 = 0$.

Taking $h = k = \alpha = \gamma = 1$, it follows from Lemma 11 that (4.4) is soluble for $a = d = 1$ unless

$$\beta \equiv (h/k) \qquad \mathrm{mod}(1/k, 2\alpha),$$

i.e.,

$$\pm a_5 + b_5x_4 - x_6 - x_7 - 2c_5 \equiv 0 \qquad (\mathrm{mod}\ 1). \tag{4.10}$$

Taking $x_4 = 0$ and 1, we get $b_5 \equiv 0$ (mod 1) which implies that $b_5 = 0$. Interchanging x_2 and x_4 we get $a_5 = 0$, so that (4.10) reduces to

$$2c_5 + c_6 + c_7 \equiv 0 \qquad (\mathrm{mod}\ 1).$$

By symmetry, we can say that (4.4) is soluble unless

$$a_5 = a_6 = a_7 = b_5 = b_6 = b_7 = 0$$

and

$$\begin{aligned}
&\text{(i)} & 2c_5 + c_6 + c_7 &\equiv 0 \qquad (\mathrm{mod}\ 1)\\
&\text{(ii)} & 2c_6 + c_5 + c_7 &\equiv 0 \qquad (\mathrm{mod}\ 1)\\
&\text{(iii)} & 2c_7 + c_5 + c_6 &\equiv 0 \qquad (\mathrm{mod}\ 1).
\end{aligned}$$

From (i), (ii), and (iii) we get $c_5 \equiv c_6 \equiv c_7$ (mod 1) and $4c_i \equiv 0$ (mod 1) for $i = 5, 6, 7$. Thus $(c_5, c_6, c_7) = (0, 0, 0)$, $\pm(1/4, 1/4, 1/4)$ or $(1/2, 1/2, 1/2)$.

Now we can suppose that

$$\begin{aligned}
Q &= (x_1 + a_2x_2 + a_4x_4)\,x_2 + (x_3 + b_4x_4)\,x_4\\
&\quad - (x_5^2 + x_6^2 + x_7^2 + x_5x_6 + x_5x_7 + x_6x_7)\\
&= (x_1 + \cdots)\,x_2 + (x_3 + \cdots)\,x_4 - (x_5 + x_6/2 + x_7/2)^2\\
&\quad - 3(x_6 + x_7/3)^2/4 - 2x_7^2/3.
\end{aligned}$$

Suppose first that $b_4 = 0$.

By symmetry, $c_3 = 0$, and $a_4 = 0$ by Lemma 4.

Choose $x_1 = c_1$, $x_2 = 0$, $x_3 = x_4 = 1$, $|x_7| \leqslant 1/2$, $|x_6 + x_7/3| \leqslant 1/2$, and $|x_5 + x_6/2 + x_7/2| \leqslant 1/2$, then

$$\begin{aligned} 0 &< 1 - 1/4 - 3/16 - 1/16 \\ &\leqslant Q = 1 - (x_5 + \cdots)^2 - 3(x_6 + \cdots)^2/4 - 2x_7^2/3 \leqslant 1. \end{aligned}$$

Therefore $0 < Q < 1$ unless

$$|x_7| = |x_6 + x_7/3| = |x_5 + x_6/2 + x_7/2| = 0,$$

i.e., $c_5 = c_6 = c_7 = 0$.

Thus, in case $b_4 = 0$, (3.18) is soluble unless $c_5 = c_6 = c_7 = 0$.

Now suppose that $b_4 \neq 0$.

Choose $(x_3, x_5, x_6, x_7) \equiv (c_3, c_5, c_6, c_7) \pmod 1$ arbitrarily, $x_2 = \pm 1$. Write $x_1 = x + c_1$ and $x_4 = y$. Then (3.18) is soluble if we can find integers x and y such that

$$0 < \pm x + \beta y \pm |b_4|\, y^2 + v < 1, \tag{4.11}$$

where $\beta = \pm a_4 + x_3$ and v is some constant. Since $0 < |b_4| \leqslant 1/2$, we have

$$|1/2 - |b_4|\,| + 1/2 = 1 - |b_4| < 1.$$

Therefore, taking $h = 1/2$, $k = 1$, $\alpha = |b_4|$, $\gamma = 1$, it follows from Lemma 11 that (4.11) is soluble except perhaps when $b_4 = 1/2$ and

$$\beta \equiv (h/k) \qquad \operatorname{mod}(1/k, 2\alpha),$$

i.e.,

$$\pm a_4 + x_3 \equiv (1/2) \qquad (\operatorname{mod} 1),$$

i.e.,

$$(a_4, c_3) = (0, 1/2) \text{ or } (1/2, 0).$$

If $(b_4, c_3) = (1/2, 0)$, choose $x_1 = c_1$, $x_2 = x_3 = 0$ and $(x_5, x_6, x_7) = (c_5, c_6, c_7)$. Then choose $x_4 = 1$ if $(c_5, c_6, c_7) = (0, 0, 0)$ or $\pm(1/4, 1/4, 1/4)$ and $x_4 = 2$ if $(c_5, c_6, c_7) = (1/2, 1/2, 1/2)$ so that

$$0 < Q = x_4^2/2 - (c_5^2 + c_6^2 + c_7^2 + c_5c_6 + c_5c_7 + c_6c_7) \leqslant 1/2 < 1.$$

Thus, (3.18) is soluble in case $(b_4, c_3) = (1/2, 0)$. If $(b_4, c_3) = (1/2, 1/2)$, then choose $x_1 = c_1$, $x_2 = 0$, $x_3 = 1/2$, $x_4 = 1$, $|x_7| \leqslant 1/2$, $|x_6 + x_7/3| \leqslant 1/2$, and $|x_5 + x_6/2 + x_7/2| \leqslant 1/2$, so that

$$\begin{aligned} 0 &< 1 - 1/4 - 3/16 - 1/16 \leqslant Q \\ &= 1/2 + 1/2 - (x_5 + \cdots)^2 - 3(x_6 + \cdots)^2/4 - 2x_7^2/3 \leqslant 1. \end{aligned}$$

Therefore, it again follows that $0 < Q < 1$ except perhaps when

$$|x_7| = |x_6 + x_7/3| = |x_5 + x_6/2 + x_7/2| = 0,$$

i.e., $c_5 = c_6 = c_7 = 0$.

Thus, (3.18) is soluble except perhaps when

$$(b_4, c_3, a_4, c_5, c_6, c_7) = (0, ..., 0) \text{ or } (1/2, 1/2, 0, ..., 0).$$

By a similar argument, interchanging x_1, x_2 with x_3, x_4, we see that (3.18) is soluble except when

$$(a_2, c_1, a_4, c_5, c_6, c_7) = (0, ..., 0) \text{ or } (1/2, 1/2, 0, ..., 0).$$

It now follows that (3.18) is soluble unless Q is equivalent to F_1, F_2, F_3, or F_4 and $(c_1, ..., c_7)$ is equivalent to the point L_1, L_2, L_3, or L_4, respectively, where

$$\begin{aligned} F_1 &= x_1x_2 + (x_3 + x_4/2)\,x_4 - \phi(x_5, x_6, x_7), \\ &\qquad L_1 = (0, 0, 1/2, 0, ..., 0), \\ F_2 &= x_1x_2 + x_3x_4 - \phi(x_5, x_6, x_7), \\ &\qquad L_2 = (0, ..., 0), \\ F_3 &= (x_1 + x_2/2)\,x_2 + (x_3 + x_4/2)\,x_4 - \phi(x_5, x_6, x_7), \\ &\qquad L_3 = (1/2, 0, 1/2, 0, ..., 0), \\ F_4 &= (x_1 + x_2/2)\,x_2 + x_3x_4 - \phi(x_5, x_6, x_7), \\ &\qquad L_4 = (1/2, 0, ..., 0), \end{aligned}$$

and

$$\phi(x_5, x_6, x_7) = x_5^2 + x_6^2 + x_7^2 + x_5x_6 + x_5x_7 + x_6x_7.$$

It is easy to see that (1.1) is soluble in each case with the sign of equality being necessary. Moreover, $F_2 = Q_2$ and F_1, F_3, and F_4 are equivalent to ρQ_1 $\rho > 0$. This is so because

$$\begin{aligned} 2(x + y/2)\,y - z^2 &= 2xy + y^2 - z^2 \\ &= (y + x)^2 - x^2 - z^2 \sim y^2 - x^2 - z^2 \\ &= 2(y - x)(z + y) - (x - y - z)^2 \sim 2yz - x^2. \end{aligned}$$

One can also check that the points L_1, L_3, L_4 go to $(1/2, ..., 1/2)$ under the corresponding transformation. So up to equivalence and positive multiples, we have two forms Q_1 and Q_2 for which equality is necessary in (1.1) for the corresponding points.

This completes the proof of the lemma.

Remark 1. We can suppose now that for $a < 1/2$, $m \leqslant 2$ and $a + d \leqslant 1$ so that $1 \geqslant a + d \geqslant d/3 + d = 4d/3$, i.e., $d \leqslant 3/4$.

Thus, we are left with the following cases:

(i) $m = 2$, $a < 1/2$, $a + d \leqslant 1$, $d \leqslant 3/4$.

(ii) $m = 2$, $a = 1/2$, $d \leqslant 1$, $h_6 = h_7 = b_5 = 0$, and $(a_5, c_5) = (0, 1/2)$ or $(1/2, 0)$.

(iii) $m = 1$, $a < 1/2$, $a + d \leqslant 1$, $d \leqslant 3/4$.

(iv) $m = 1$, $a = 1/2$, $d \leqslant 1$, $h_6 = h_7 = b_5 = a_5 = 0$, and $c_5 = 1/2$.

5. $m = 2$

5.1. $a < 1/2$, $a + d \leqslant 1$, $d \leqslant 3/4$.

Relation (3.18) can be written as

$$\begin{aligned} 0 < & -a(x_5 + h_6 x_6 + h_7 x_7 + (1/2a)\, a_5 x_2 + (b_5 x_4)/a)^2 \\ & + (x_1 + a_2' x_2 + \cdots)\, x_2 + 2(x_3 + b_4' x_4 + \cdots)\, x_4 - \psi(x_6, x_7) < d. \end{aligned} \tag{5.1}$$

Now $27 \geqslant (d/a)^3 \geqslant 4d^3/d^7 = 4/d^4 \geqslant 4(4/3)^4 > 8$ [by (3.10)] so that

$$2 < d/a \leqslant 3.$$

By Lemma 7, (5.1) is soluble if we can solve

$$1/4 < (1/a)(x_1 + \cdots)\, x_2 + (2/a)(x_3 + \cdots)\, x_4 - (1/a)\, \psi(x_6, x_7) < (d/a) + 1,$$

i.e.,

$$0 < (x_1 + \cdots)\, x_2 + 2(x_3 + \cdots)\, x_4 - Ax_6^2 - Bx_6 x_7 - Cx_7^2 - a/4 < d + 3a/4. \tag{5.2}$$

By (3.13) and (3.16)

$$A \leqslant \sqrt{d^7/6a} \leqslant d^3/\sqrt{2},$$

so that

$$(d+3a/4)/A \geqslant 5d/4A \geqslant 5\sqrt{2}\, d/4d^3 \geqslant 20\sqrt{2}/9 > 3.$$

Let N be an integer satisfying

$$N < [d+3a/4]/A \leqslant N+1,$$

then $N \geqslant 3$.

Now (5.2) can be written as

$$\begin{aligned} 0 < -A(x_6 + \cdots)^2 + (x_1 + a_2'' x_2 + \cdots)\, x_2 \\ + 2(x_3 + b_4'' x_4 + \cdots)\, x_4 - \lambda x_7^2 - a/4 < d + 3a/4. \end{aligned} \tag{5.3}$$

By Lemma 7, (5.3) is soluble if we can solve

$$\begin{aligned} 0 < (x_1 + \cdots)\, x_2 + 2(x_3 + \cdots)\, x_4 - \lambda x_7^2 \\ -(a+A)/4 < d + 3a/4 + A(N^2-1)/4. \end{aligned} \tag{5.4}$$

Now

$$\begin{aligned} d + 3a/4 + (N^2-1)\, A/4 \geqslant (d+3a/4)(N+3)/4 \geqslant 5d(N+3)/16 \\ \geqslant 15d/8 \geqslant (15/8)(4/27)^{1/4} > 1. \end{aligned}$$

Choose $(x_3, x_4, x_7) = (c_3, c_4, c_7)$, $x_2 = 1$, and $x_1 \equiv c_1 \pmod 1$ in such a way that

$$\begin{aligned} 0 < (x_1 + \cdots)\, x_2 + 2(x_3 + \cdots)\, x_4 - \lambda x_7^2 - (a+A)/4 \\ \leqslant x_2 = 1 < d + 3a/4 + (N^2-1)\, A/4, \end{aligned}$$

so that (5.4) and hence, (3.18) is soluble in case $m = 2$ and $a < 1/2$.

5.2. $a = 1/2$.

By (3.10),

$$1/2 = a \leqslant (d^7/m^2)^{1/3} = (d^7/4)^{1/3} \Rightarrow d^7 \geqslant 1/2 \Rightarrow d \geqslant 0.905....$$

By Remark 1(ii), we have $h_6 = h_7 = b_5 = 0$ and $(a_5, c_5) = (0, 1/2)$ or $(1/2, 0)$, so that (3.18) can be written as

$$0 < (x_1 + \cdots)\, x_2 + 2(x_3 + \cdots)\, x_4 - x_5^2/2 - Ax_6^2 - Bx_6x_7 - Cx_7^2 < d. \tag{5.5}$$

By (3.13) and (3.15) we have

$$1/2 = a \leqslant A \leqslant (d^7/6a)^{1/2} = (d^7/3)^{1/2}. \tag{5.6}$$

LEMMA 14. *Relation* (5.5) *is soluble unless* $A=C=1/2$, $B=0$, $d=1$, $b_6=b_7=0$, $c_4=0$ *or* $1/2$, $(a_6, c_6)=(0, 1/2)$ *or* $(1/2, 0)$, *and* $(a_7, c_7)=(0, 1/2)$ *or* $(1/2, 0)$.

Proof. Choose $(x_3, x_4, x_5, x_7) \equiv (c_3, c_4, c_5, c_7) \pmod 1$ arbitrarily, $x_2=\pm 1$. Write $x_1=x+c_1$ and $x_6=y+c_6$. Then (5.5) is soluble if we can find integers x and y such that

$$0 < \pm x + \beta y - Ay^2 + v < d, \tag{5.7}$$

where $\beta = \pm a_6 + 2b_6 x_4 - 2Ac_6 - Bx_7$ and v is some constant. Now, by (5.6) we have

$$|1/2 - A| + 1/2 = A \leqslant (d^7/3)^{1/2} < d \qquad (\because d \leqslant 1).$$

Therefore, taking $h=1/2$, $k=1$, $\alpha=A$, $\gamma=d$, it follows from Lemma 11 that (5.7) is soluble unless $A=1/2$ and

$$\beta \equiv (h/k) \qquad \mathrm{mod}(1/k, 2\alpha),$$

i.e.,

$$\pm a_6 + 2b_6 x_4 - 2Ac_6 - Bx_7 \equiv 1/2 \qquad (\mathrm{mod}\ 1).$$

Taking $x_7=c_7$ and $1+c_7$, we get $B \equiv 0 \pmod 1$. Since $0 \leqslant B \leqslant A = 1/2$, therefore $B=0$, so that

$$1/4 = A^2 \leqslant AC = \det \psi = \Delta/a = 2\Delta = 8\,|D| = d^7/4 \leqslant 1/4$$
$$\Rightarrow 1/2 = A = C \qquad \text{and} \qquad d=1.$$

Now we can suppose that

$$Q = (x_1 + \cdots)\, x_2 + 2(x_3 + b_4 x_4 + b_6 x_6 + b_7 x_7)\, x_4 - x_5^2/2 - x_6^2/2 - x_7^2/2.$$

Since $(a_5, c_5)=(0, 1/2)$ or $(1/2, 0)$ and $b_5=0$, therefore, by symmetry, we can suppose that $b_6=b_7=0$, $(a_6, c_6)=(0, 1/2)$ or $(1/2, 0)$, and $(a_7, c_7)=(0, 1/2)$ or $(1/2, 0)$.

Choose $x_1=c_1$, $x_2=0$, $(x_5, x_6, x_7)=(c_5, c_6, c_7)$. If $c_4 \neq 0$ or $1/2$, then choose $0<|x_4|<1/2$. Then $x_3 \equiv c_3 \pmod 1$ satisfying

$$0 < (x_1 + \cdots)\, x_2 + 2(x_3 + \cdots)\, x_4 - x_5^2/2 - x_6^2/2 - x_7^2/2 \leqslant 2\,|x_4| < 1$$

and, hence, (3.18) is soluble unless $c_4=0$ or $1/2$. This completes the proof of the lemma.

Remark 2. We can now suppose that $a=A=C=1/2$, $B=0$, $d=1$, $h_6=h_7=b_5=b_6=b_7=0$, $(a_5, c_5)=(0, 1/2)$ or $(1/2, 0)$, $(a_6, c_6)=(0, 1/2)$ or $(1/2, 0)$, $(a_7, c_7)=(0, 1/2)$ or $(1/2, 0)$, and $c_4=0$ or $1/2$.

LEMMA 15. *Relation* (3.18) *is soluble if* $c_4=0$.

Proof. Choose $(x_3, ..., x_7) \equiv (c_3, ..., c_7) \pmod 1$ arbitrarily, $x_2 = \pm 1$. Choose $x_1 \equiv c_1 \pmod 1$ satisfying

$$0 < Q \leqslant |x_2| = 1.$$

If $0<Q<1$, then (3.18) is soluble, otherwise we can suppose that $Q \equiv 0 \pmod 1$. That is,

$$(x_1+a_2x_2+\cdots)\,x_2+2(x_3+b_4x_4)\,x_4-x_5^2/2-x_6^2/2-x_7^2/2 \equiv 0 \pmod 1.$$

Choose $x_4=0$, $(x_1, x_5, x_6, x_7)=(c_1, c_5, c_6, c_7)$, $x_2= \pm 1$ to get

$$\pm c_1+a_2+a_5c_5+a_6c_6+a_7c_7-(c_5^2+c_6^2+c_7^2)/2 \equiv 0 \pmod 1.$$

But $a_5c_5=a_6c_6=a_7c_7=0$ by Remark 2. Therefore,

$$\pm c_1+a_2-(c_5^2+c_6^2+c_7^2)/2 \equiv 0 \pmod 1.$$

These congruences imply that

$$2a_2 \equiv c_5^2+c_6^2+c_7^2 \pmod 1 \tag{5.8}$$

and

$$c_1 \equiv -a_2+(c_5^2+c_6^2+c_7^2)/2 \pmod 1. \tag{5.9}$$

Since a_2 is a value of the form, therefore for $0<|a_2|<1/3$, (3.18) follows from Lemma 12. So $a_2=0$ or $|a_2| \geqslant 1/3$. In case $a_2=0$, by Lemma 5, $a_4=a_5=a_6=a_7=0$.

In the various cases that arise from (5.8), (5.9), and symmetry, (3.18) is soluble as shown in Table I. This proves the lemma.

TABLE I

c_5	c_6	c_7	a_5	a_6	a_7	a_2	c_1	$(x_1, ..., x_7)$	$Q(x_1, ..., x_7)$
0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$(-\frac{1}{2}, 2, c_3, 0, -1, 1, 1)$	$\frac{1}{2}$
$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{8}$	$\frac{1}{2}$	$(\frac{3}{2}, 2, c_3, 0, \frac{3}{2}, 0, 0)$	$\frac{3}{8}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	$\frac{3}{8}$	0	$(0, 2, c_3, 0, \frac{1}{2}, \frac{1}{2}, \frac{3}{2})$	$\frac{1}{8}$

LEMMA 16. *Relation* (3.18) *is soluble if* $c_4 = 1/2$ *unless* $Q \sim \rho Q_3$, $\rho > 0$, *and* $(c_1, \ldots, c_7) = (1/2, \ldots, 1/2)$ *in which case equality is necessary in* (1.1).

Proof. Choose $(x_1, x_2, x_6, x_7) \equiv (c_1, c_2, c_6, c_7) \pmod 1$ arbitrarily, $x_4 = \pm 1/2$. Write $x_3 = x + c_3$ and $x_5 = y + c_5$. Then (3.18) is soluble if we can find integers x and y such that

$$0 < \pm x + \beta y - y^2/2 + v < 1, \tag{5.10}$$

where

$$\beta = \pm a_5 x_2 - c_5 \quad \text{and} \quad v \text{ is some constant.}$$

Taking $h = 1/2$, $k = 1$, $\alpha = 1/2$, $\gamma = 1$, it follows from Lemma 11 that (5.10) is soluble except perhaps when

$$a_5 x_2 - c_5 \equiv (1/2) \pmod 1.$$

Taking $x_2 = 0$ and 1, we get $a_5 \equiv 0 \pmod 1$, which implies that $a_5 = 0$ and hence $c_5 = 1/2$.

By symmetry, we can suppose that (3.18) is soluble unless $a_6 = a_7 = 0$ and $c_6 = c_7 = 1/2$, so that

$$Q = (x_1 + a_2 x_2 + a_4 x_4)\, x_2 + 2(x_3 + b_4 x_4)\, x_4 - (x_5^2 + x_6^2 + x_7^2)/2.$$

Choose $x_1 = c_1$, $x_2 = 0$, $x_4 = \pm 1/2$, and $x_5 = x_6 = x_7 = 1/2$, then choose $x_3 \equiv c_3 \pmod 1$ in such a way that

$$0 < Q = \pm x_3 + b_4/2 - 3/8 \leqslant 1.$$

If $0 < Q < 1$ then we are through, otherwise $Q \equiv 0 \pmod 1$, i.e.,

$$\pm c_3 + b_4/2 - 3/8 \equiv 0 \pmod 1,$$
$$\Rightarrow b_4 \equiv -1/4 \pmod 1,$$

i.e., $b_4 = -1/4$ and, hence, $c_3 = 1/2$. Thus, (3.18) is soluble unless $(b_4, c_3) = (-1/4, 1/2)$.

Now choose $(x_1, x_5, x_6, x_7) \equiv (c_1, c_5, c_6, c_7) \pmod 1$ arbitrarily, $x_4 = \pm 1/2$. Write $x_3 = x + c_3$ and $x_2 = y$. Suppose $a_2 \neq 0$. Then (3.18) is soluble if we can find integers x and y such that

$$0 < \pm x + \beta y \pm |a_2|\, y^2 + v < 1, \tag{5.11}$$

where

$$\beta = x_1 \pm a_4/2 \qquad \text{and} \qquad \nu \text{ is some constant.}$$

Since

$$|1/2 - |a_2|\,| + 1/2 = 1 - |a_2| < 1,$$

taking $h = 1/2$, $k = 1$, $\alpha = |a_2|$, $\gamma = 1$, it follows from Lemma 11 that (5.11) is soluble except perhaps when $a_2 = 1/2$ and

$$x_1 \pm a_4/2 \equiv 1/2 \pmod 1.$$

So $(a_4, c_1) = (0, 1/2)$. That is, (3.18) is soluble unless $a_2 = 0$ or

$$(a_2, a_4, c_1) = (1/2, 0, 1/2).$$

In case $a_2 = 0$, then $a_4 = 0$ by Lemma 4 and therefore, by symmetry, $c_1 = 0$.

It now follows that (3.18) is soluble unless $Q \sim F_1$ or F_2 and $(c_1, ..., c_7)$ is equivalent to the points L_1 and L_2, respectively, where

$$F_1 = x_1 x_2 + 2(x_3 - x_4/4)\,x_4 - (x_5^2 + x_6^2 + x_7^2)/2 \qquad \text{and}$$
$$L_1 = (0, 0, 1/2, ..., 1/2),$$
$$F_2 = (x_1 + x_2/2)\,x_2 + 2(x_3 - x_4/4)\,x_4 - (x_5^2 + x_6^2 + x_7^2)/2 \qquad \text{and}$$
$$L_2 = (1/2, 0, 1/2, ..., 1/2).$$

It is easy to see that F_1 and F_2 are equivalent to ρQ_3 and L_1, L_2 go to $(1/2, ..., 1/2)$ under the corresponding transformation. Moreover, equality in (1.1) is necessary in each of these cases, since

$$\begin{aligned} 8F_1(x_1, x_2, x_3 + 1/2, x_4 + 1/2, ..., x_7 + 1/2) &\\ = 8x_1 x_2 + 4(2x_3 + 1)(2x_4 + 1) - (2x_4 + 1)^2 - (2x_5 + 1)^2 &\\ - (2x_6 + 1)^2 - (2x_7 + 1)^2 \equiv 0 \pmod 8, & \end{aligned}$$

for integers x_i. This proves the lemma.

6. $m = 1$

6.1. $a < 1/2$, $a + d \leqslant 1$, $d \leqslant 3/4$.

Relation (3.18) can be written as

$$\begin{aligned} 0 < -(x_5 + \cdots)^2 + (1/a)(x_1 + a_2' x_2 + \cdots)\,x_2 &\\ + (1/a)(x_3 + b_4' x_4 + \cdots)\,x_4 - (1/a)\,\psi(x_6, x_7) < d/a & \qquad (6.1) \end{aligned}$$

for some rationals a_i' and b_j'.

Now $d/a \geqslant d/d^{7/3} = (1/d^4)^{1/3} > (4/3)^{4/3} > 1$ and $d/a \leqslant 3$ [by (3.16)], so that $1 < d/a \leqslant 3$. Let K be an integer satisfying $K < d/a \leqslant K+1$, so that $K=1$ or 2. By Lemma 7, (6.1) is soluble if we can solve

$$1/4 < (1/a)(x_1 + \cdots)\,x_2 + (1/a)(x_3 + \cdots)\,x_4 - (1/a)\,\psi(x_6, x_7) < d/a + K^2/4. \tag{6.2}$$

Case (i). $K=2$, i.e.,

$$2 < d/a \leqslant 3. \tag{6.3}$$

Now (6.2) can be written as

$$0 < (x_1 + a_2'x_2 + a_4'x_4 + a_6'x_6 + a_7'x_7)\,x_2 + (x_3 + b_4'x_4 + b_6'x_6 + b_7'x_7)\,x_4 - Ax_6^2 - Bx_6x_7 - Cx_7^2 - a/4 < d + 3a/4. \tag{6.4}$$

Further, we can suppose that $-1/2 < a_i' \leqslant 1/2$ and $-(1/2) < b_j' \leqslant 1/2$, on replacing x_1 by $x_1 + m_2x_2 + m_4x_4 + m_6x_6 + m_7x_7$ and x_3 by $x_3 + n_4x_4 + n_6x_6 + n_7x_7$ where m_i and n_i are suitable integers.

Now (6.4) can be rewritten as

$$0 < -(x_6 + \cdots)^2 + (1/A)(x_1 + a_2''x_2 + \cdots)\,x_2 + (1/A)(x_3 + b_4''x_4 + \cdots)\,x_4 - (1/A^2)(\det\psi)\,x_7^2 - a/4A < (d + 3a/4)/A. \tag{6.5}$$

By (3.13) and (6.3), for $d \leqslant 3/4$,

$$\frac{d + 3a/4}{A} \geqslant \frac{5d}{4A} \geqslant \frac{5d}{4}\sqrt{\frac{3a}{2d^7}} \geqslant \frac{5}{4\sqrt{2}\,d^2} \geqslant \frac{20}{9\sqrt{2}} > 1.$$

Let N be the integer satisfying $N < (d + 3a/4)/A \leqslant N+1$, so that $N \geqslant 1$. By Lemma 7, (6.5) is soluble if we can solve

$$1/4 < (1/A)(x_1 + \cdots)\,x_2 + (1/A)(x_3 + \cdots)\,x_4 - (1/A^2)(\det\psi)\,x_7^2 - a/4A < (d + 3a/4)/A + N^2/4,$$

i.e.,

$$A/4 + a/4 < (x_1 + \cdots)\,x_2 + (x_3 + \cdots)\,x_4 - (1/A)(\det\psi)\,x_7^2 < d + a + N^2A/4. \tag{6.6}$$

Since

$$(x_1 + \cdots)\, x_2 + (x_3 + \cdots)\, x_4 - (1/A)(\det \psi)\, x_7^2$$

is a zero form of the determinant $-(\det \psi/16A) = -\Delta/16aA = -|D|/aA = -(d^7/32aA)$, it follows from Lemma 5 that (6.6) is soluble if

$$d + 3a/4 + [(N^2-1)/4]\, A > 2(d^7/32aA)^{1/5} = (d^7/aA)^{1/5},$$

i.e., if

$$(d + 3a/4 + [(N^2-1)/4]\, A)^5\, aAd^{-7} > 1. \tag{6.7}$$

Now

$$\begin{aligned}
&[d + 3a/4 + \{(N^2-1)/4\}\, A]^5\, aAd^{-7} \\
&\quad \geqslant [(d+3a/4)(N+3)/4)]^5\, [(d+3a/4)/(N+1)]\, ad^{-7} \\
&\quad \geqslant (d + d/4)^6\, [(N+3)^5/4^5(N+1)]\, (d/3)\, d^{-7} \\
&\quad = 5^6(N+3)^5/(3\,.\,4^{11}(N+1)).
\end{aligned}$$

Since $f(N) = (N+3)^5/(N+1)$ is an increasing function of N, for $N \geqslant 2$, $f(N) \geqslant f(2) = 5^5/3 > 3\,.\,4^{11}/5^6$, so that for $N \geqslant 2$, (6.7) is satisfied and, hence, (6.6) is soluble.

Now we are left with $N = 1$, i.e.,

$$1 < (d + 3a/4)/A \leqslant 2, \tag{6.8}$$

which implies that

$$5d/4 \leqslant (d + 3a/4) \leqslant 2A \leqslant 2\sqrt{(2d^7/3a)} \leqslant 2\sqrt{2}\, d^3 \qquad \text{(by (3.13) and (6.3))}$$

and, hence

$$d^2 \geqslant 5/8\,\sqrt{2} \Rightarrow d \geqslant 0.66\ldots. \tag{6.9}$$

LEMMA 17. *Relation* (6.4) *is soluble except perhaps when* $A = 1/2$, $B = 0$, $a_6' = b_6' = 0$, *and* $c_6 = 1/2$.

Proof. Choose $(x_3, x_4, x_7) \equiv (c_3, c_4, c_7)$ (mod 1) arbitrarily, $x_2 = \pm 1$. Write $x_1 = x + c_1$ and $x_6 = y + c_6$. Then (6.4) is soluble if we can find integers x and y such that

$$0 < \pm x + \beta y - Ay^2 + \nu < d + 3a/4, \tag{6.10}$$

where

$$\beta = \pm a_6' + b_6' x_4 - 2Ac_6 - Bx_7 \qquad \text{and} \qquad v \text{ is some constant.}$$

Taking $h = 1/2$, $k = 1$, $\alpha = A$, and $\gamma = d + 3a/4$, it follows from Lemma 11 that (6.10) is soluble for $A \neq 1/2$ if

$$|1/2 - A| + 1/2 < d + 3a/4. \tag{6.11}$$

For $A > 1/2$, (6.11) is satisfied by (6.8). For $A < 1/2$, by (6.8), (6.3), and (6.9), we have

$$\begin{aligned} A + d + 3a/4 &\geqslant (d + 3a/4)/2 + (d + 3a/4) = 3(d + 3a/4)/2 \\ &\geqslant 3(d + d/4)/2 = 15d/8 > 1, \end{aligned}$$

so that (6.11) is satisfied.

For $A = 1/2$ with $h = 1/2$, $k = 1$, $\alpha = 1/2$, $\gamma = d + 3a/4$, it follows from Lemma 11 that (6.10) is soluble except perhaps when

$$\beta \equiv h/k \qquad \operatorname{mod}(1/k, 2\alpha),$$

i.e.,

$$\pm a_6' + b_6' x_4 - c_6 - Bx_7 \equiv 1/2 \qquad (\operatorname{mod} 1) \tag{6.12}$$

for all choices of x_4 and x_7.

Taking $x_7 = c_7$ and $1 + c_7$ we get $B \equiv 0 \pmod 1$. Since $0 \leqslant B \leqslant A = 1/2$, we have $B = 0$.

Taking $x_4 = 0$ and 1, we get $b_6' \equiv 0 \pmod 1$. Since $-1/2 < b_6' \leqslant 1/2$, we have $b_6' = 0$. Interchanging the roles of x_2 and x_4, we get $a_6' = 0$. Substituting $a_6' = b_6' = B = 0$ in (6.12) we get

$$c_6 \equiv 1/2 \pmod 1 \Rightarrow c_6 = 1/2.$$

Therefore, it follows that (6.10) and, hence, (6.4) is soluble unless $A = 1/2$, $B = 0$, $a_6' = b_6' = 0$, and $c_6 = 1/2$.

LEMMA 18. *Relation* (6.4) *is soluble for* $A = 1/2$, $B = 0$, $a_6' = b_6' = 0$, $c_6 = 1/2$ *unless* $C = 1/2$, $b_7' = a_7' = 0$, *and* $c_7 = 1/2$.

Proof. We have

$$C/2 = AC = \det \psi = \Delta/a = 16\,|D|/a = d^7/2a,$$

so that

$$1/2 = A \leqslant C = d^7/a \leqslant 3d^6 \qquad (\because d/3 \leqslant a).$$

Choose $(x_3, x_4, x_6) \equiv (c_3, c_4, c_6) \pmod 1$ arbitrarily, $x_2 = \pm 1$. Write $x_1 = x + c_1$ and $x_7 = y + c_7$. Then (6.4) is soluble if we can find integers x and y such that

$$0 < \pm x + \beta y - Cy^2 + v < d + 3a/4, \tag{6.13}$$

where

$$\beta = \pm a'_7 + b'_7 x_4 - 2Cc_7 \qquad \text{and} \qquad v \text{ is some constant.}$$

Since $1/2 \leqslant C \leqslant 3d^6$, $a \geqslant d/3$, and $d \leqslant 3/4$, we have

$$|1/2 - C| + 1/2 = C \leqslant 3d^6 < 5d/4 \leqslant d + 3a/4.$$

Therefore, taking $h = 1/2$, $k = 1$, $\alpha = C$, $\gamma = d + 3a/4$, it follows from Lemma 11 that (6.13) is soluble except perhaps when $C = 1/2$ and

$$\beta \equiv h/k \qquad \mathrm{mod}(1/k, 2\alpha),$$

i.e.,

$$\pm a'_7 + b'_7 x_4 - c_7 \equiv 1/2 \qquad \pmod 1.$$

Proceeding as before, we get $a'_7 = b'_7 = 0$ and $c_7 = 1/2$.

Thus, (6.13) and, hence, (6.4) is soluble unless $A = C = 1/2$, $B = 0$, $a'_6 = a'_7 = b'_6 = b'_7 = 0$, and $c_6 = c_7 = 1/2$.

Remark 3. We can suppose that $A = C = 1/2$, $B = 0 = a'_6 = b'_6 = a'_7 = b'_7$, and $c_6 = c_7 = 1/2$. Therefroe, (6.4) can be written as

$$\begin{aligned} 0 &< (x_1 + a'_2 x_2 + a'_4 x_4)\, x_2 + (x_3 + b'_4 x_4)\, x_4 - (1/2)(x_6^2 + x_7^2) - a/4 \\ &< d + 3a/4, \end{aligned} \tag{6.14}$$

where $d^7 = 32\,|D| = a/2$; so that

$$2d^7 = a \geqslant d/3 \Rightarrow d \geqslant 0.741\ldots. \tag{6.15}$$

LEMMA 19. *Relation* (6.14) *is soluble unless* $(b'_4, c_3) = (0, 0)$, $(1/2, 0)$, *or* $(1/2, 1/2)$.

Proof. Suppose first that $b'_4 \neq 0$. Choose $(x_3, x_6, x_7) \equiv (c_3, c_6, c_7)$ $\pmod 1$ arbitrarily, $x_2 = \pm 1$. Write $x_1 = x + c_1$ and $x_4 = y$, then, (6.14) is soluble if we can find integers x and y such that

$$0 < \pm x + \beta y \pm |b'_4|\, y^2 + v < d + 3a/4, \tag{6.16}$$

where

$$\beta = \pm a_4' + x_3 \qquad \text{and} \qquad \nu \text{ is some constant.}$$

Let $h = 1/2$, $k = 1$, $\alpha = |b_4'|$, and $\gamma = d + 3a/4$. Then it follows from Lemma 11 that (6.16) is soluble for $b_4' \neq 1/2$ if

$$|1/2 - |b_4'|\ | + 1/2 < d + 3a/4,$$

i.e., if

$$1 < |b_4'| + d + 3a/4.$$

Now suppose that

$$|b_4'| + d + 3a/4 \leqslant 1 \qquad \text{for} \quad b_4' \neq 0 \text{ or } 1/2. \tag{6.17}$$

For $b_4' \neq 0$ or $1/2$, (6.14) can be written as

$$\begin{aligned} 0 &< b_4'[x_4 + (a_4'/2b_4')\, x_2 + x_3/2b_4']^2 \\ &\quad + (x_1 + \lambda_2 x_2 + \lambda_3 x_3)\, x_2 - x_3^2/4b_4' - x_6^2/2 - x_7^2/2 - a/4 \\ &< d + 3a/4. \end{aligned} \tag{6.18}$$

By (6.17), (6.3), and (6.15)

$$(d + 3a/4)/|b_4'| \geqslant (d + 3a/4)/(1 - d - 3a/4) > 10.$$

Let M be the integer satisfying

$$M < (d + 3a/4)/|b_4'| \leqslant M + 1, \qquad \text{then } M \geqslant 10.$$

By Lemmas 6 and 7, (6.18) is soluble if we can solve

$$\begin{aligned} 0 &< (x_1 + \cdots)\, x_2 - x_3^2/4b_4' - x_6^2/2 - x_7^2/2 - a/4 + \lambda \\ &< d + 3a/4 + [(M^2 - 1)/4]\, |b_4'|, \end{aligned} \tag{6.19}$$

where

$$\lambda = \begin{cases} -|b_4'|/4 & \text{if } b_4' \text{ is negative} \\ M^2 b_4'/4 & \text{if } b_4' \text{ is positive.} \end{cases}$$

By (6.3) and (6.15), for $M \geqslant 10$

$$\begin{aligned} (d + 3a/4) + (M^2 - 1)\, |b_4'|/4 &\geqslant (d + 3a/4)(M + 3)/4 \\ &\geqslant 5d(M + 3)/16 \geqslant 65d/16 > 1, \end{aligned}$$

so that (6.19) is soluble by taking $(x_3, x_6, x_7) \equiv (c_3, c_6, c_7)$, $x_2 = 1$, and then $x_1 \equiv c_1 \pmod 1$ satisfying

$$0 < (x_1 + \cdots)\, x_2 - x_3^2/4b_4' - x_6^2/2 - x_7^2/2 - a/4 + \lambda \leqslant x_2 = 1$$
$$< d + 3a/4 + (M^2 - 1)\, |b_4'|/4.$$

Thus, we are left with $b_4' = 0$ or $1/2$.

In case $b_4' = 0$ and $c_3 \neq 0$, choose $x_1 = c_1$, $x_2 = 0$, $(x_6, x_7) = (c_6, c_7)$, $x_3 \equiv c_3 \pmod 1$ satisfying $0 < |x_3| \leqslant 1/2$ and then choose x_4 in such a way that

$$0 < (x_1 + \cdots)\, x_2 + x_3 x_4 - x_6^2/2 - x_7^2/2 - a/4 \leqslant |x_3| \leqslant 1/2 < d + 3a/4.$$

Therefore, (6.14) is soluble for $b_4' = 0$ unless $c_3 = 0$. In case $b_4' = 1/2$, it follows from Lemma 11, with $h = 1/2$, $k = 1$, $\alpha = 1/2$, $\gamma = d + 3a/4$, that (6.16) is soluble unless

$$\beta \equiv h/k \qquad \mathrm{mod}(1/k,\, 2\alpha),$$

i.e.,

$$\pm a_4' + x_3 \equiv 1/2 \qquad (\mathrm{mod}\ 1),$$

so that $2c_3 \equiv 0 \pmod 1$, i.e., $c_3 = 0$ or $1/2$.

This completes the proof of the lemma.

LEMMA 20. *Relation* (6.14) *is soluble for* $(b_4', c_3) = (0, 0)$, $(1/2, 1/2)$, *and* $(1/2, 0)$.

Proof. Choose $x_1 = c_1$, $x_2 = 0$, $x_6 = x_7 = 1/2$, $x_4 = 1$, so that $(x_1 + \cdots)\, x_2 + (x_3 + b_4' x_4)\, x_4 - x_6^2/2 - x_7^2/2 - a/4 = x_3 + b_4' - 1/4 - a/4$. On taking $x_3 = c_3$ if $b_4' = 1/2$ and $x_3 = 1$ if $(b_4', c_3) = (0, 0)$ it is easy to see that

$$0 < x_3 + b_4' - 1/4 - a/4 < 3/4 - a/4 < d + 3a/4,$$

and, hence, (6.14) is soluble.

Case (ii). $K = 1$, i.e., $1 < d/a \leqslant 2$, so that

$$d/2 \leqslant a \leqslant d^{7/3} \Rightarrow d^4 \geqslant 1/8.$$

Moreover

$$1 \geqslant a + d \geqslant d/2 + d = 3d/2 \Rightarrow d \leqslant 2/3. \tag{6.20}$$

Now (6.2) can be written as

$$0<(x_1+\cdots)x_2+(x_3+\cdots)x_4-Ax_6^2-Bx_6x_7-Cx_7^2-a/4<d$$

or

$$0<-A(x_6+\cdots)^2+(x_1+a_2''x_2+\cdots)x_2+(x_3+b_4''x_4+\cdots)x_4$$
$$-(\det\psi)x_7^2/A-a/4<d. \tag{6.21}$$

We have

$$3d/8\leqslant 3a/4\leqslant A\leqslant\sqrt{2d^7/3a}<2d^3/\sqrt{3} \qquad \text{(by (3.13) and (3.15))}$$

so that

$$8/3\geqslant d/A\geqslant\sqrt{3}d/2d^3\geqslant 9\sqrt{3}/8>1.$$

Let N be the integer satisfying $N<d/A\leqslant N+1$, then $N=1$ or 2. By Lemma 7, (6.21) is soluble if we can solve

$$A/4<(x_1+\cdots)x_2+(x_3+\cdots)x_4-(\det\psi)x_7^2/A-a/4<d+N^2A/4. \tag{6.22}$$

LEMMA 21. *Relation* (6.22) *is soluble for* $N=2$.

Proof. For $N=2$, (6.22) can be written as

$$0<(x_1+\cdots)x_2+(x_3+\cdots)x_4-(\det\psi)x_7^2/A-(a+A)/4<d+3A/4. \tag{6.23}$$

Let $t=\det\psi/A=\Delta/aA=16|D|/aA=d^7/2aA\leqslant 8d^5/3$ ($\because d/2\leqslant a$, $d/A\leqslant 8/3$) therefore

$$(d+3A/4)/t\geqslant 41d/32t\geqslant(123/256d^4)\geqslant 123(3/2)^4/256>2.$$

Let M be the integer satisfying

$$M<(d+3A/4)/t\leqslant M+1, \qquad \text{then} \quad M\geqslant 2.$$

By Lemma 7, (6.23) is soluble if we can solve

$$t/4<(x_1+\cdots)x_2+(x_3+\cdots)x_4-(a+A)/4<d+3A/4+M^2t/4$$

or

$$0<(x_1+\cdots)x_2+(x_3+\cdots)x_4-(a+A+t)/4<d+3A/4+(M^2-1)t/4. \tag{6.24}$$

Now

$$41d/32 \leqslant d + 3A/4 \leqslant (M+1)t \leqslant [8(M+1)d^5]/3, \tag{6.25}$$

so that

$$d^4 \geqslant 123/256(M+1)$$

and

$$\begin{aligned}(d+3A/4+(M^2-1)\,t/4)^4 &\geqslant ((d+3A/4)(M+3)/4)^4 \geqslant (41d(M+3)/128)^4\\ &\geqslant 3(41/128)^5\,(M+3)^4/2(M+1)\\ &= f(M) \geqslant f(2) > 1.\end{aligned}$$

Now choose $(x_3, x_4) = (c_3, c_4)$, $x_2 = 1$, and $x_1 \equiv c_1 \pmod 1$ satisfying

$$\begin{aligned}&0 < (x_1 + \cdots)\,x_2 + (x_3 + \cdots)\,x_4 - (a+A+t)/4 \leqslant x_2 = 1\\ &\quad < d + 3a/4 + (M^2-1)\,t/4.\end{aligned}$$

Therefore, (6.24) and, hence, (6.22) is soluble.

Remark 4. We are left with $N=1$, i.e., $1 < d/A \leqslant 2$, so that using (3.13) we have

$$\begin{aligned}&d/2 \leqslant A \leqslant \sqrt{2d^7/3a} \leqslant 2d^3/\sqrt{3} \qquad (\because d/a \leqslant 2)\\ &\Rightarrow d \geqslant 0.658.\end{aligned}$$

LEMMA 22. *Relation* (3.18) *is soluble except perhaps when* $a = 1/3$, $h_6 = 0$ *or* $1/2$ *and* $h_7 = 0$ *or* $1/2$.

Proof. Choose $(x_3, x_4, x_6, x_7) \equiv (c_3, c_4, c_6, c_7) \pmod 1$ arbitrarily, $x_2 = \pm 1$. Write $x_1 = x + c_1$, $x_5 = y + c_5$. Then (3.18) is soluble if we can find integers x and y such that

$$0 < \pm x + \beta y - ay^2 + v < d, \tag{6.26}$$

where

$\beta = \pm a_5 + b_5 x_4 - 2ah_6 x_6 - 2ah_7 x_7 - 2ac_5$ and v is some constant.

Let $h = k = 3$ and $\alpha = a$. Consider the inequality

$$|3 - 9a| + 1/2 < d. \tag{6.27}$$

For $a \geqslant 1/3$, (6.27) is satisfied, since

$$9a \leqslant 9(1-d) < 5/2+d \qquad \text{for} \quad d \geqslant 0.658... > 13/20.$$

For $a < 1/3$, (6.27) is satisfied, since

$$9a+d \geqslant 9d/2+d = 11d/2 > 7/2 \qquad \text{for} \quad d \geqslant 0.658... > 7/11.$$

Therefore, it follows from Lemma 11 that (6.26) is soluble except perhaps when $a = 1/3$ and

$$\beta \equiv h/k \qquad [\bmod(1/k, 2\alpha)],$$

i.e.,

$$\pm a_5 + b_5 x_4 - 2h_6 x_6/3 - 2h_7 x_7/3 - 2c_5/3 \equiv 0 \qquad (\bmod 1/3). \tag{6.28}$$

Taking $x_6 = c_6$ and $1+c_6$, we get $2h_6/3 \equiv 0$ (mod 1/3), i.e., $2h_6 \equiv 0$ (mod 1) $\Rightarrow h_6 = 0$ or $1/2$.

Similarly, $h_7 = 0$ or $1/2$. This proves the lemma.

LEMMA 23. *Relation* (3.18) *is soluble for* $a = 1/3$ *if*

(i) $h_6 = 1/2$.

(ii) $h_6 = 0$ *unless* $A = B = C = 1/3$ *and* $h_7 = 0$.

Proof. Relation (3.18) can be written as

$$\begin{aligned} 0 < (x_1 + \cdots)\, x_2 + (x_3 + \cdots)\, x_4 + (x_5 + h_6 x_6 + h_7 x_7)^2/3 - A x_6^2 \\ - B x_6 x_7 - C x_7^2 < d. \end{aligned}$$

Choose $(x_3, x_4, x_5, x_7) \equiv (c_3, c_4, c_5, c_7)$ (mod 1) arbitrarily, $x_2 = \pm 1$. Write $x_1 = x + c_1$ and $x_6 = y + c_6$. Then the above inequality is soluble if we can find integers x and y such that

$$0 < \pm x + \beta y - (A + h_6^2/3)\, y^2 + v < d, \tag{6.29}$$

where $\beta = \pm a_6 + b_6 x_4 - 2(A + h_6^2/3)\, c_6 - 2h_6(x_5 + h_7 x_7)/3 - B x_7$ and v is some constant. We have, by (3.13) and (3.14),

$$\begin{aligned} 1/3 = a \leqslant A + h_6^2/3 &\leqslant \sqrt{2d^7/3a} + 1/12 \\ &= \sqrt{2d^7} + 1/12 < 1/2 \qquad (\because d \leqslant 2/3). \end{aligned}$$

Taking $h=1/2$, $k=1$, $\alpha=A+h_6^2/3$, $\gamma=d$, it follows from Lemma 11 that (6.29) is soluble if

$$|1/2-(A+h_6^2/3)|+1/2<d,$$

i.e., if

$$1<A+h_6^2/3+d. \tag{6.30}$$

In case $h_6=1/2$, since $N=1$, we have $A\geqslant d/2$, so that

$$A+d+h_6^2/3\geqslant d/2+d+1/12=3d/2+1/12>1 \qquad (\because d\geqslant 0.658...),$$

so that (6.30) is satisfied for $h_6=1/2$. In case $h_6=0$, if (6.30) is satisfied then (6.29) is soluble, otherwise we can suppose that $A+d\leqslant 1$. Moreover, in this case $1/3=a\leqslant A$ [by (3.15)], so that $|3-9A|+1/2<d$, since $d>13/20$.

Therefore, taking $h=k=3$, $\alpha=A$, and $\gamma=d$ in Lemma 11, it follows that (6.28) is soluble unless $h_6=0$, $A=1/3$, and

$$\beta\equiv h/k \qquad (\mathrm{mod}\ 1/k,\ 2\alpha),$$

i.e.,

$$\pm a_6+b_6x_4-2c_6/3-Bx_7\equiv 0 \qquad (\mathrm{mod}\ 1/3).$$

Taking $x_7=c_7$ and $1+c_7$, we get $B\equiv 0$ (mod 1/3). Since $0\leqslant B\leqslant A=1/3$, we have $B=0$ or $1/3$.

In case $B=0$, we have

$$C/3=AC=\det\psi=\Delta/a=3\Delta=48\,|D|=3d^7/2,$$

$$1/3=A\leqslant C=9d^7/2,$$

$$d^7\geqslant 2/27, \qquad \text{i.e.,} \qquad d\geqslant 0.689..., \text{ a contradiction, since}$$

$$d\leqslant 2/3, \qquad \text{by (6.20)}.$$

Therefore, $B=1/3$.

Now we can suppose that $a=A=B=1/3$ and $h_6=0$, so that

$$Q=(x_1+\cdots)x_2+(x_3+\cdots)x_4-(x_5+h_7x_7)^2/3-x_6^2/3-x_6x_7/3-Cx_7^2,$$

where

$$C/3=AC\leqslant 4\Delta/3a=4\Delta=64\,|D|=2d^7,$$

so that

$$1/3 = A \leqslant C \leqslant 6d^7$$

and

$$1/3 \leqslant C + h_7^2/3 \leqslant 6d^7 + 1/12 < 1/2 \qquad (\because d \leqslant 2/3).$$

Moreover,

$$1 < C + h_7^2/3 + d \qquad \text{for} \quad h_7 = 1/2.$$

Therefore, proceeding as above, it can be seen that (3.18) is soluble except when $C = 1/3$ and $h_7 = 0$ and the lemma follows.

LEMMA 24. *Relation* (3.18) *is soluble if* $a = A = B = C = 1/3$ *and* $h_6 = h_7 = 0$.

Proof. In this case (3.18) can be written as

$$\begin{aligned} 0 < &-(x_6 + x_7/2 + \cdots)^2/3 + (x_1 + a_2' x_2 + \cdots) x_2 + (x_3 + b_4' x_4 + \cdots) x_4 \\ &- x_5^2/3 - x_7^2/4 < d. \end{aligned}$$

By Lemma 7, this inequality is soluble if we can solve

$$1/4 < 3(x_1 + \cdots) x_2 + 3(x_3 + \cdots) x_4 - x_5^2 - 3x_7^2/4 < 3d + 1/4,$$

since $1 < 3d \leqslant 2$. That is, if we can solve

$$0 < (x_1 + \cdots) x_2 + (x_3 + \cdots) x_4 - x_5^2/3 - x_7^2/4 - 1/12 < d,$$

i.e.,

$$\begin{aligned} 0 < &-(x_7 + \cdots)^2/4 + (x_1 + a_2'' x_2 + \cdots) x_2 + (x_3 + b_4'' x_4 + \cdots) x_4 \\ &- x_5^2/3 - 1/12 < d. \end{aligned} \tag{6.31}$$

Since $2 < 4d \leqslant 8/3$, by Lemma 7, (6.31) is soluble if we can solve

$$0 < (x_1 + \cdots) x_2 + (x_3 + \cdots) x_4 - x_5^2/3 - 1/12 - 1/16 < d + 3/16. \tag{6.32}$$

Choose $(x_3, x_4) \equiv (c_3, c_4) \pmod 1$ arbitrarily, $x_2 = \pm 1$. Write $x_1 = x + c_1$, $x_5 = y + c_5$. Then (6.32) is soluble if we can find integers x and y such that

$$0 < \pm x + \beta y - y^2/3 + v < d + 3/16. \tag{6.33}$$

Taking $h=1/2$, $k=1$, $\alpha=1/3$, and $\gamma=d+3/16$, it follows from Lemma 11 that (6.33) is soluble if

$$|1/2-1/3|+1/2<d+3/16,$$

i.e., if $d>2/3-3/16=0.47...$ which is true.

This proves the lemma.

6.2. $a=1/2$.

By Remark 1(iv), we have $d\leqslant 1$, $h_6=h_7=b_5=a_5=0$, and $c_5=1/2$. Moreover,

$$1/2=a\leqslant A<\sqrt{4d^7/3} \qquad [\text{by (3.13) and (3.15)}]$$
$$\Rightarrow d^7\geqslant 3/16 \Rightarrow d\geqslant 0.787....$$

LEMMA 25. *For* $a=1/2$, $m=1$, (3.18) *is soluble unless* $b_6=b_7=0$ *and* $(b_4, c_3)=(0,0)$, $(1/2,0)$, *or* $(1/2,1/2)$.

Proof. b_4 is a value of the form. If $0<|b_4|<d/3$ then (1.1) is soluble with strict inequality by Lemma 12. Therefore $b_4=0$ or $d/3\leqslant|b_4|\leqslant 1/2$. Suppose first that $b_4\neq 0$. Choose $(x_3, x_5, x_6, x_7)\equiv(c_3, c_5, c_6, c_7)$ (mod 1) arbitrarily, $x_2=\pm 1$. Write $x_1=x+c_1$ and $x_4=y$. Then (3.18) is soluble if we can find integers x and y such that

$$0<\pm x+\beta y\pm|b_4|\,y^2+v<d, \tag{6.34}$$

where

$$\beta=\pm a_4+x_3+b_6x_6+b_7x_7 \qquad \text{and} \qquad v \text{ is some constant.}$$

Since $d/3\leqslant|b_4|\leqslant 1/2$ and $d>3/4$, we have

$$|1/2-|b_4||+1/2<d.$$

Therefore, taking $h=1/2$, $k=1$, $\alpha=|b_4|$, and $\gamma=d$, it follows from Lemma 11 that (6.34) is soluble unless $|b_4|=1/2$, i.e., $b_4=1/2$ and

$$\beta\equiv h/k \qquad \mathrm{mod}(1/k, 2\alpha),$$

i.e.,

$$\pm a_4+x_3+b_6x_6+b_7x_7\equiv 1/2 \qquad (\mathrm{mod}\ 1).$$

Taking $x_6=c_6$ and $1+c_6$, we get $b_6\equiv 0$ (mod 1), i.e., $b_6=0$. Similarly, $b_7=0$ and hence $\pm a_4+c_3\equiv 1/2$ (mod 1) which imply that $c_3=0$ or $1/2$, i.e., (3.18) is soluble unless $(b_4, c_3)=(1/2,0)$ or $(1/2,1/2)$ and $b_6=b_7=0$.

If $b_4 = 0$ then $b_5 = b_6 = b_7 = 0$ and therefore, by symmetry $c_3 = 0$. This completes the proof of the lemma.

Remark 5. For $a = 1/2$, $m = 1$ we can suppose that $b_6 = b_7 = 0$ and $(b_4, c_3) = (0, 0)$, $(1/2, 0)$, or $(1/2, 1/2)$.

LEMMA 26. *For $a = 1/2$, $m = 1$, (3.18) is soluble except perhaps when*

(i) $A = 1/2$, $B = 0$, *and* $c_6 = 1/2$.

(ii) $A = 1$, $B = 0$ *or* 1.

Proof. Choose $(x_1, x_2, x_5, x_7) \equiv (c_1, c_2, c_5, c_7) \pmod 1$ arbitrarily, $x_4 = \pm 1$. Write $x_3 = x + c_3$ and $x_6 = y + c_6$. Then (3.18) is soluble if we can find integers x and y such that

$$0 < \pm x + \beta y - A y^2 + v < d, \tag{6.35}$$

where

$$\beta = -2Ac_6 - Bx_7 \qquad \text{and} \qquad v \text{ is some constant.}$$

Taking $h = 1/2$, $k = 1$, $\alpha = A$, and $\gamma = d$, it follows from Lemma 11 that (6.35) is soluble for $A \neq 1/2$ if

$$|1/2 - A| + 1/2 < d,$$

i.e., if

$$A < d \qquad (\because A > 1/2).$$

Otherwise, suppose that

$$d \leqslant A. \tag{6.36}$$

Now take $h = k = 1$ so that (6.35) is soluble by Lemma 11 for $A \neq 1$ if

$$|1 - A| + 1/2 < d. \tag{6.37}$$

For $A < 1$, (6.37) is satisfied since $A \geqslant d > 3/4$. For $A > 1$, (6.37) is satisfied, since

$$A - d < (4d^7/3)^{1/2} - d < 1/2 \qquad \text{for} \quad d \leqslant 1.$$

Therefore, (3.18) is soluble unless $A = 1/2$ or 1. For $A = 1/2$ and 1, taking $h = \alpha = A$, $k = 1$, $\gamma = d$, it follows from Lemma 11, that (6.35) and hence, (3.18) is soluble unless

$$\beta \equiv h/k \qquad \mathrm{mod}(1/k, 2\alpha),$$

i.e.,

$$-2Ac_6 - Bx_7 \equiv A \qquad (\text{mod } 1).$$

Taking $x_7 = c_7$ and $1 + c_7$, we get $B \equiv 0$ (mod 1). Since $0 \leqslant B \leqslant A$, we have $B = 0$, $c_6 = 1/2$ in case $A = 1/2$ and $B = 0$ or 1 in case $A = 1$, and the result follows.

LEMMA 27. *For $a = 1/2$, (3.18) is soluble for $A = 1/2$, $B = 0$, and $c_6 = 1/2$.*

Proof. In this case

$$Q = (x_1 + \cdots) x_2 + (x_3 + b_4 x_4) x_4 - x_5^2/2 - x_6^2/2 - Cx_7^2,$$

where

$$C/2 = AC = \Delta/a = 2\Delta = 32\,|D| = d^7 \leqslant 1,$$

so that

$$1/2 = A \leqslant C = 2d^7 \leqslant 2.$$
$$\Rightarrow d^7 \geqslant 1/4 \Rightarrow d \geqslant 0.82....$$

By Remark 5, $(b_4, c_3) = (0, 0)$ or $(1/2, 1/2)$ or $(1/2, 0)$. Choose $x_1 = c_1$, $x_2 = 0$, $x_6 = 1/2$, $|x_7| \leqslant 1/2$, so that

$$Q = (x_3 + b_4 x_4) x_4 - x_5^2/2 - 1/8 - Cx_7^2.$$

Then choosing $(x_3, x_4, x_5) = (1, 1, 1/2)$, $(0, 2, 3/2)$ or $(1/2, 1, 1/2)$ according to whether $(b_4, c_3) = (0, 0)$, $(1/2, 0)$, or $(1/2, 1/2)$, respectively, we see that

$$0 < 3/4 - 1/2 \leqslant 3/4 - C/4 \leqslant Q = 3/4 - Cx_7^2 \leqslant 3/4 < d.$$

Therefore, (3.18) is soluble.

LEMMA 28. *For $a = 1/2$, (3.18) is soluble for $A = 1$ and $B = 0$ or 1.*

Proof. By (3.13)

$$1 = A \leqslant (4d^7/3)^{1/2} \Rightarrow d^7 \geqslant 3/4 \Rightarrow d > 0.9....$$

Moreover,

$$1 = A \leqslant C = AC \leqslant 4\Delta/3a = 4d^7/3 \leqslant 4/3.$$

$$\begin{aligned} Q &= (x_1 + \cdots) x_2 + (x_3 + b_4 x_4) x_4 - x_5^2/2 - x_6^2 - Bx_6 x_7 - Cx_7^2 \\ &= (x_1 + \cdots) x_2 + (x_3 + b_4 x_4) x_4 - x_5^2/2 - (x_6 + \lambda x_7)^2 \\ &\quad - (C - \lambda^2) x_7^2, \end{aligned}$$

where $\lambda = 0$ or $1/2$ according to whether $B = 0$ or 1. By Remark 5, we have $(b_4, c_3) = (0, 0)$, $(1/2, 0)$, or $(1/2, 1/2)$. Choose $x_1 = c_1$, $x_2 = 0$, so that

$$Q = (x_3 + b_4 x_4)\, x_4 - x_5^2/2 - (x_6 + \lambda x_7)^2 - (C - \lambda^2)\, x_7^2.$$

Then choosing $|x_7| \leqslant 1/2$, $|x_6 + \lambda x_7| \leqslant 1/2$, and $(x_3, x_4, x_5) = (1, 1, 1/2)$, $(0, 2, 3/2)$, or $(1/2, 1, 1/2)$ according to whether $(b_4, c_3) = (0, 0)$, $(1/2, 0)$ or $(1/2, 1/2)$, respectively, we see that

$$0 < 5/8 - C/4 \leqslant Q = 7/8 - (x_6 + \lambda x_7)^2 - (C - \lambda^2)\, x_7^2 \leqslant 7/8 < d.$$

Therefore, (3.18) is soluble.

Lemmas 12–28 complete the proof for rational forms and hence of the Theorem for all forms.

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